double-diffusive convection from a line source in a porous medium, in agreement with the findings in ref. [2] for natural convection driven by temperature gradients alone.

Due to the prohibitive complexity of the algebraic calculations, higher-order solutions in Ra were impossible to obtain.

4. THE PRESENCE OF A VERTICAL INSULATED WALL IN THE VICINITY OF THE SOURCE

Based on the previous results it is possible to shed light on the effect of the presence of a vertical insulated wall on the flow field induced by the line source. Assume that the insulated vertical wall constitutes the y_*-axis of an (x_*-y_*) Cartesian coordinate system and that the line source is located at $x_* = d$, $y_* = 0$. This arrangement is equivalent to an arrangement consisting of two line sources positioned at $y_* = 0, x_* = \pm d$, with the vertical wall removed [2]. The zeroth-order solution corresponds to the case of no fluid motion and it is reported in ref. [9]. Hence, it is not repeated here for brevity. As explained in ref. [2], due to the linearity of the momentum equation the solution for ψ_1 is simply the superposition of solutions for line sources at $x = \pm 1, y = 0$. In this part of the study d was used as the reference length for the non-dimensionalization. The final expression for ψ_1 reads

$$\psi_1 = \frac{\tau^{1/2}}{4\pi}(S_+ + S_-) \tag{16}$$

where

$$\begin{aligned} S_\pm = {} & \frac{2\tau^{1/2}(x\pm 1)}{(x\pm 1)^2+y^2}\left\{\exp\left[-\frac{(x\pm 1)^2+y^2}{4\tau}\right]-1\right\} \\ & -\frac{x\pm 1}{2\tau^{1/2}}\int_{[(x\pm 1)^2+y^2]/4\tau}^{\infty}\frac{\exp(-\xi)}{\xi}\,\mathrm{d}\xi \\ & -\frac{A}{B}\left\{\frac{2\tau^{1/2}(x\pm 1)}{(x\pm 1)^2+y^2}\,\frac{1}{B}\left(\exp\left[-B^2\,\frac{(x\pm 1)^2+y^2}{4\tau}\right]-1\right)\right. \\ & \left.-\frac{B(x\pm 1)}{2\tau^{1/2}}\int_{B^2[(x\pm 1)^2+y^2]/4\tau}^{\infty}\frac{\exp(-\xi)}{\xi}\,\mathrm{d}\xi\right\}. \end{aligned} \tag{17}$$

The first-order equations for c_1 and T_1 are non-linear, therefore, it is not possible to obtain c_1 and T_1 by superposition. The streamline pattern $\psi_1/(1-A) = \text{const}$ for $\tau = 1$, $B = 1$ was identical to the streamline pattern in ref. [2] where the concentration-gradient-induced buoyancy was neglected ($A = 0$). Basically, the presence of the wall flattens the streamlines in the wall vicinity. For an illustration of this effect ref. [2] is recommended.

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A numerical solution to moving boundary problems—application to melting and solidification

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1. INTRODUCTION

A LARGE number of technically important problems are classified as moving boundary. Heat transfer problems with a phase change, litospheric movement according to plate tectonics, gas–solid reactions occurring in a moving reaction zone are all of the moving boundary type. For various kinds of such problems there are solutions available [1]. For the case that the thermal conductivity varies linearly with temperature Cho and Sunderland [2] presented an exact solution; Voller and Cross [3] have investigated the same problem in two dimensions. Cheung *et al* [4] presented numerical solutions for a finite slab with internal heat generation.

Analytical solutions, although very convenient, can only be applied to very specific cases. In situations where physical properties depend on system variables the analytical solutions are impossible. Problems with various complexities and boundary conditions can be analyzed numerically using superfast computers.

An approximation commonly adopted in numerical approach is that the phase boundary movement and also the changes in transient quantities occur at a constant rate in a

NOMENCLATURE

a, b	proportionality constants
C	specific heat capacity [J kg^{-1} K^{-1}]
d, h	spatial step size in liquid and solid, respectively, [m]
H	latent heat of solidification [J kg^{-1}]
i	position integer; y/h for solid, $m+(L-Y)/d$ for liquid
k	thermal conductivity [W m^{-1} K^{-1}]
m, n	grid points in the solid and total number, respectively
t	time [s]
T, X	old and new temperatures, respectively [°C]
Y	thickness of solidified layer [m]
y	space co-ordinate.

Greek symbols

α	thermal diffusivity [m^2 s^{-1}]
ρ	density [kg m^{-3}]
δ	step size in time [s].

Subscripts

i, m, n	grid point i, interface and insulated boundary, respectively
l, s	liquid and solid, respectively
0	initial or previous
w	wall.

given time step. In order to calculate the values of the variables at a new time step an equation of the following form is generally used

(new value) = (old value) + (rate of change) × (time step). (1)

The rate of change is taken either at the previous or the next time node. The approximation holds if the time step is sufficiently small. However this requires more computer time and may cause instabilities. Since rate of change various from time $= t$ to $t+\delta$, it should be some average rate containing explicit and implicit terms. In this paper we intend to deal with the problems using averaged expressions and also introducing a coordinate system with a fixed number of grid points in both phases. This system accommodates sudden changes in variables, thus allowing smaller spatial steps when changes are fast and larger steps when changes are slow. In addition, a changing time step is used, making it easier to follow crucial changes very closely. Otherwise a sudden change would be considered to last for a finite time segment δ, whereas it lasts only for $\lim \delta \to 0$.

A physical problem is chosen and solved by explicit and average rates. Solution with average rate is found to be far more stable allowing much greater time steps than explicit methods. The problem is described below.

2. PHYSICAL PROBLEM AND FORMULATION

Consider one-dimensional solidification of molten steel. Liquid occupies the space $0 < y < L$ with a temperature T_0, initially with $T_0 > T_m$ where T_m is the solidification temperature; the liquid is insulated at one end. At time = 0 and $y = 0$ the temperature is set to a low value T_w with $T_w < T_m$. A solid layer starts forming and the phase-change boundary will move. We can write the conservation of heat in various regions as

solid:
$$\rho_s C_s \frac{\partial T_s}{\partial t} = \frac{\partial}{\partial y}\left(k_s \frac{\partial T_s}{\partial y}\right), \quad 0 \leqslant y < Y \qquad (2)$$

liquid:
$$\rho_1 C_1 \frac{\partial T_1}{\partial t} = \frac{\partial}{\partial y}\left(k_1 \frac{\partial T_1}{\partial y}\right), \quad Y < y \leqslant L \qquad (3)$$

phase-change boundary:
$$\rho_1 H \frac{\mathrm{d}Y}{\mathrm{d}t} = k_1 \left.\frac{\partial T_1}{\partial y}\right|_{y=Y} - k_s \left.\frac{\partial T_s}{\partial y}\right|_{y=Y}. \qquad (4)$$

Equation (4) is written for conduction dominating convection in the liquid. This is true if the liquid and solid densities are equal. The initial and boundary conditions which can be substituted by others are given below.

$$T = T_0, \quad Y = 0, \quad t \leqslant 0 \qquad (5a)$$
$$T_s = T_w, \quad y = 0 \qquad (5b)$$
$$T_s = T_1 = T_m, \quad y = Y \qquad (5c)$$
$$k_1 \left.\frac{\partial T}{\partial y}\right|_{y=L} = 0, \quad y = L. \qquad (5d)$$

3. SOLUTION TECHNIQUE

The solid and liquid regions are divided into fixed number of grid points. Thus, a coordinate system is adopted where each grid point moves with a different amount, the ends being fixed. The partial time derivative of temperature can be written in terms of a gradient which is moving at a velocity of $\mathrm{d}y/\mathrm{d}t$ as

$$\frac{\partial T}{\partial t} = \frac{\mathrm{d}T}{\mathrm{d}t} - \left(\frac{\partial T}{\partial y}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right). \qquad (6)$$

Let $i = 1$ represent the grid point at the cooling surface; $i = m$, interface; $i = n$ the insulated surface. Let X_i be the temperature at time $= t+\delta$. Using the Crank–Nicolson technique various derivatives can be written as follows:

Solid region: i = 2, m − 1

$$\left(\frac{\partial^2 T}{\partial y^2}\right)_i = 0.5\left(\frac{T_{i+1}-2T_i+T_{i-1}}{h_0^2} + \frac{X_{i+1}-2X_i+X_{i-1}}{h^2}\right), \qquad (7)$$

$$\left(\frac{\mathrm{d}T}{\mathrm{d}t}\right)_i = \frac{X_i - T_i}{\delta}, \qquad (8)$$

$$\left(\frac{\partial T}{\partial y}\right)_i = 0.25\left(\frac{T_{i+1}-T_{i-1}}{h_0} + \frac{X_{i+1}-X_{i-1}}{h}\right), \qquad (9)$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_i = \frac{(Y-Y_0)(i-1)}{(m-1)\delta}. \qquad (10)$$

Using equations (7)–(10), equation (2) can be written at i as

$$\left[\frac{(i-1)(Y-Y_0)}{4Y} - \frac{\alpha_s\delta}{2h^2}\right]X_{i-1} + \left(1+\frac{\alpha_s\delta}{h^2}\right)X_i - \left[\frac{(i-1)(Y-Y_0)}{4Y} + \frac{\alpha_s\delta}{2h^2}\right]X_{i+1}$$
$$= T_i + \frac{(T_{i+1}-T_{i-1})(Y-Y_0)(i-1)}{4Y_0} + \alpha_s\delta\frac{(T_{i+1}-2T_i+T_{i-1})}{2h_0^2}. \qquad (11)$$

This can be written in a matrix form including all the grid points $i = 2, m-1$ as

$$
\begin{bmatrix}
1+\dfrac{\alpha_s\delta}{h^2} & \searrow & \dfrac{(i-1)(Y_0-Y)}{4Y}-\dfrac{\alpha_s\delta}{2h^2} \searrow \\
\dfrac{(i-1)(Y-Y_0)}{4Y}-\dfrac{\alpha_s\delta}{2h^2} \searrow & & \searrow
\end{bmatrix} \mathbf{X}_s
$$

$$
= \begin{bmatrix}
T_2+\dfrac{(T_3-T_w)(Y-Y_0)}{4Y_0}+\dfrac{\alpha_s\delta}{2h_0^2}(T_w-2T_2+T_3)+\left[\dfrac{\alpha_s\delta}{2h^2}-\dfrac{(Y-Y_0)}{4Y}\right]T_w \\
\downarrow \\
T_i+\dfrac{(T_{i+1}-T_{i-1})(Y-Y_0)(i-1)}{4Y_0}+\dfrac{\alpha_s\delta}{2h_0^2}(T_{i-1}-2T_i+T_{i+1}) \\
\downarrow \\
T_{m-1}+\dfrac{(m-2)(Y-Y_0)}{4Y_0}(T_m-T_{m-2})+\dfrac{\alpha_s\delta}{2h_0^2}(T_{m-2}-2T_{m-1}+T_m) \\
+\left[\dfrac{(m-2)(Y_0-Y)}{4Y}+\dfrac{\alpha_s\delta}{h^2}\right]T_m
\end{bmatrix}. \tag{12}
$$

Liquid region: i = m + 1, n

The derivatives in equations (3) and (6) are written in a manner similar to that in the solid region. The rate of movement of the grid point i in the liquid region which corresponds to equation (10) can be written as

$$\left(\frac{dy}{dt}\right)_i = (Y-Y_0)\frac{(n-i)}{(n-m)\delta}. \tag{13}$$

Then equation (3) becomes for $i = m+1, n$

$$
\left[\frac{(Y-Y_0)(n-i)}{4(L-Y)}-\frac{\alpha_l\delta}{2d^2}\right]X_{i-1}+\left(1+\frac{\alpha_l\delta}{d^2}\right)X_i
-\left[\frac{(Y-Y_0)(n-i)}{4(L-Y)}+\frac{\alpha_l\delta}{2d^2}\right]X_{i+1}
= T_i+\frac{(Y-Y_0)(n-i)}{4(L-Y_0)}(T_{i+1}-T_{i-1})
+\frac{\alpha_l\delta}{2d_0^2}(T_{i+1}-2T_i+T_{i-1}). \tag{14}
$$

The set of equations obtained from equation (14) can be written in matrix form as

$$
\begin{bmatrix}
1+\dfrac{\alpha_l\delta}{d^2} & \searrow & \dfrac{(Y_0-Y)(n-i)}{4(L-Y)}-\dfrac{\alpha_l\delta}{2d^2} \searrow \\
\dfrac{(Y-Y_0)(n-i)}{4(L-Y)}-\dfrac{\alpha_l\delta}{2d^2} \searrow & \searrow & -\dfrac{\alpha_l\delta}{d^2}
\end{bmatrix} \mathbf{X}_l
$$

$$
= \begin{bmatrix}
T_{m+1}+\dfrac{(Y-Y_0)(n-m-1)}{4(L-Y_0)}(T_{m+2}-T_m)+\dfrac{\alpha_l\delta}{2d_0^2}(T_{m+2}-2T_{m+1}+T_m)+\left[\dfrac{\alpha_l\delta}{2d^2}-\dfrac{Y-Y_0}{4(L-Y)}(n-i)\right]T_m \\
\downarrow \\
T_i+\dfrac{(Y-Y_0)(n-i)}{4(L-Y_0)}(T_{i+1}-T_{i-1})+\dfrac{\alpha_l\delta}{2d_0^2}(T_{i+1}-2T_i+T_{i-1}) \\
\downarrow \\
T_n+\dfrac{\alpha_l\delta}{d_0^2}(T_{n-1}-T_n)
\end{bmatrix}. \tag{15}
$$

In order to solve equations (12) and (15) we need to obtain Y from equation (4) as

$$Y = Y_0 + \frac{\alpha_l \delta}{2\rho_l H}\left[k_l\left(\frac{T_{m+1}-T_m}{d_0} + \frac{X_{m+1}-T_m}{d}\right) - k_s\left(\frac{T_m - T_{m-1}}{h_0} + \frac{T_m - X_{m-1}}{h}\right)\right]. \quad (16)$$

Since equation (16) involves the new temperatures we need to solve the system of equations (12), (15) and (16) using an iterative technique. In order to start the iteration an initial value of Y is assumed and the quantities d and h are calculated. Then equations (12) and (15) are solved. Using the new temperatures an improved Y is obtained through equation (16); this is repeated until the improvement in Y is very small. Also a variable time step $\delta = \text{const}\, h^2/2\alpha_s$ is used where const is a parameter to monitor the effect of time step on the solution. Therefore, for a given const, δ is a function of time.

3. RESULTS AND DISCUSSION

The computer solutions are obtained through the implicit method [equations (12), (15) and (16)] and an explicit method for various values of const. The following data are used: $T_w = 90°C$; $T_m = 1480°C$, $T_0 = 2200°C$; ρ_s, $\rho_l = 7800$ kg m^{-3}; C_s, $C_l = 420$ J kg^{-1} K^{-1}; $k_s = 34$ W m^{-1} K^{-1}; $k_l = 17$ W m^{-1}K^{-1}; $H = -250000$ J kg^{-1}; $L = 1$ m, $n = 20$. In order to find a basis for comparison the problem is solved by both methods using very small time steps. Both methods produce similar results with no appreciable difference. The T vs y plots are compared with approximate analytical solutions [5] and the maximum difference is found to be 1.5%. Error analyses are based on these solutions.

In problems involving the evaluation of various physical parameters depending on system variables an important means of comparison is the number of iterations necessary for a certain amount of solidification. The method requiring fewer iterations involving a certain error will use less time when complex evaluations have to be made. Figure 1 shows the plot of the number of iterations vs CPU time required. Both methods are stable beyond upper limits of iteration. Explicit method becomes unstable below about 4400 iterations (11.8 s), whereas implicit method is unstable below 100 iterations (1.8 s). Therefore, implicit method is superior as evidenced by the iteration ratio of 1 : 5.

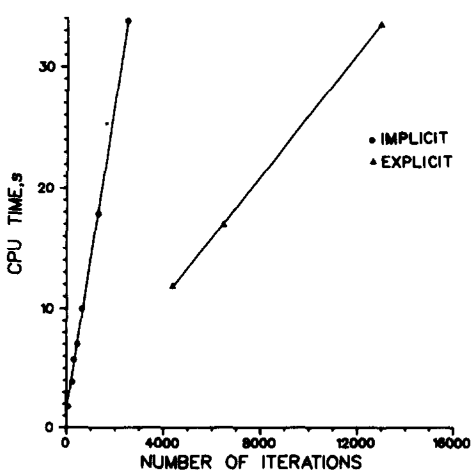

FIG. 1. CPU time vs number of iterations.

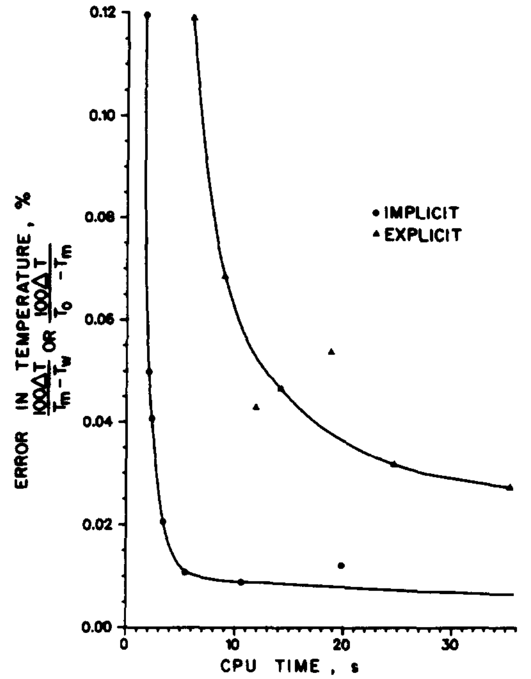

FIG. 2. Percentage maximum global error in termperature vs CPU time.

Figure 2 shows maximum error in temperature vs CPU for both methods. In the ordinate ΔT is the maximum global error in temperature of solid or liquid. Obviously at all levels of CPU implicit method produces smaller errors, that is, for any given error CPU required is smaller for the implicit method. Figure 3 shows percentage error in the time necessary to solidify 80% of liquid vs CPU. For any given error CPU is smaller in implicit method.

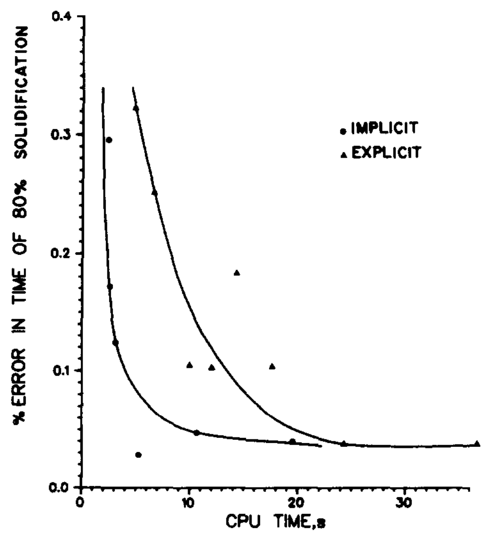

FIG. 3. Error percentage of the time for 80% solidification vs CPU time.

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Propagation of the temperature front in heat-up of an initially isothermal fluid

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1. INTRODUCTION

UNSTEADY thermal convection of an initially isothermal fluid in a closed cavity has lately received considerable attention in the literature (see, e.g. [1–5]). Most of these papers studied the transient behavior of a Boussinesq fluid as a result of impulsively imposed thermal forcings on the boundaries of the cavity. As in common technological applications, we are interested in situations in which the overall Rayleigh number, $Ra = \alpha g \Delta T h^3/\nu\kappa$, is sufficiently large to render a boundary-layer-type character. Here, α is the coefficient of volumetric expansion, g the gravity, ΔT the characteristic temperature difference, h the height of the cavity, ν the kinematic viscosity, and κ the thermal diffusivity. We are restricted to the cases for which the final state is of a gravitationally stable configuration. The Prandtl number of the fluid, $Pr \equiv \nu/\kappa$, is taken to be $O(1)$. The aspect ratio of the cavity is $O(1)$.

As was succinctly expounded in ref. [1], the dominant mechanism is the pumping by the buoyant boundary layers on the vertical walls of the container; this induces convective circulations in the inviscid core. Therefore, the decisive thermal forcing is that on the vertical walls. Consequently, the temperature adjustment in the core is accomplished principally by the convective activities rather than by diffusion.

One salient feature of the temperature evolution is the presence of the vertically propagating temperature front [4, 6]. Reference [4] examined an exemplary case when a uniform temperature gradient $\Delta T/h$ is abruptly applied to the sidewall of a vertically-mounted cylinder (radius a, height h). During the transient phase, the temperature field in the core is divided into two regions by a horizontal front. Ahead of the front, the fluid remains non-stratified, retaining the uniform temperature of the initial state; behind the front, the fluid is stratified. Reference [4] showed that the characteristic time for the front to traverse the height of the cylinder is given by the convective time scale $Ra^{1/4}N_f^{-1}$, N_f being the Brunt–Väisälä frequency in the final state, $N_f = (\alpha g \Delta T/h)^{1/2}$. It was also found that the propagation speed of the front is fairly constant over much of the cylinder depth.

To observe experimentally the front propagation described by ref. [4], it is necessary that the sidewall be made of a material of extremely high thermal conductivity. This will ensure that the fluid temperature at the inner surface of the wall is equalized to the temperature at the outer surface of the wall. The outside temperature T_e is controlled to give a desired thermal forcing for the particular experiment.

The requirement of having perfectly conducting walls poses a severe difficulty for laboratory apparatus. In order to understand more realistic systems, it is useful to inquire into the effect of finitely conducting boundaries on the front propagation. Recently, ref. [7] proposed a highly idealized model which provides a lowest-order description for the front propagation in a cylinder whose vertical sidewall has a finite thermal conductivity. The transient process is initiated by a uniform, impulsive increase in the ambient temperature. Reference [7] formulated the boundary-layer transport to determine the position of the propagating front that separates the isothermal and stratified regions. Most significantly, ref. [7] derived the characteristic time for the front as functions of the externally-controlled physical parameters.

In this note, by conducting numerical experiments we shall verify the front propagation predicted by Rahm's model [7]. Numerical solutions to the time-dependent Navier–Stokes equations were acquired. The theoretical predictions will be compared against the numerical results using different values for the sidewall thermal conductance and for Ra.

2. THE THEORETICAL MODEL

In this section, the lowest-order expressions for the front propagation will be briefly described. For full details, the reader is referred to the original paper [7].

Consider a quiescent incompressible fluid contained in a closed straight cylinder, with insulated horizontal endwalls at $z = 0$ and $z = h$, respectively. The radial and vertical coordinates are denoted by r and z. The initial state is in thermal equilibrium at uniform temperature T_0 everywhere. At $t = 0$, the temperature of the environment is suddenly raised to $T_e > T_0$, and it is maintained so thereafter. The vertical sidewall is finitely conducting, and the Newtonian heat flux condition is adopted [6–8]:

$$\frac{\partial T}{\partial r} = S(T_e - T) \quad \text{at } r = a. \tag{1}$$

Here, the thermal conductance of the sidewall is represented by S. Physically, $S = k_w/kd$, k_w and k being the thermal conductivity of the sidewall material and of the fluid, respectively, and d the thickness of the sidewall. As an example for typical laboratory situations, S is approx. 1.5 cm^{-1} if the working fluid is water and the sidewall is made of glass 1 cm thick [8].